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# Commuting charges and symmetric spaces

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## Abstract

Every classical sigma-model with target space a compact symmetric space  $G/H$  (with  $G$  classical) is shown to possess infinitely many local, commuting, conserved charges which can be written in closed form. The spins of these charges run over a characteristic set of values, playing the role of exponents of  $G/H$ , and repeating modulo an integer  $h$  which plays the role of a Coxeter number.

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# 1 Introduction

An important set of classically integrable field theories in 1+1 dimensions is provided by non-linear sigma-models with target manifold a symmetric space  $G/H$ . These have been studied for many years, particularly in connection with a one-parameter ‘dual symmetry’ which allows the construction of both local and non-local conserved quantities. Little is known about the local charges arising from this approach, however, with comparatively few examples which can be written in closed form. A comprehensive discussion, with many references, can be found in [1]. Examples of more recent work involving various models based on symmetric spaces are [2, 3, 4, 5].

Principal chiral models (PCMs) can be regarded as special cases of symmetric space models (SSMs), with target manifold a Lie group  $G$ . In [6], it was shown that other well-known local conservation laws in the PCMs, apparently different from those arising from dual symmetry, exhibit striking and hitherto unexpected properties. In particular, it was possible to construct mutually-commuting sets of charges with a characteristic pattern of spins given by the exponents of each classical group  $G$  modulo its Coxeter number. The same pattern of spins arises in affine Toda field theories, and proves central to understanding a number of their most important properties [7, 8].

In this note we will show that results similar to those of [6] can be obtained for any sigma-model based on a symmetric space  $G/H$  with  $G$  a classical group. Specifically, we will consider local conserved currents which can be written in a simple, closed form, and which lead to commuting sets of charges whose spins are related to the underlying symmetric space data. We begin by formulating the field theory and the relevant conservation laws.

Let  $g(x^\mu)$  be a field on two-dimensional Minkowski space taking values in some compact Lie group  $G$ , with Lie algebra  $\mathfrak{g}$ . Let  $H \subset G$  be some subgroup and  $\mathfrak{h} \subset \mathfrak{g}$  the corresponding Lie subalgebra. To formulate the sigma-model with target space  $G/H$ , we introduce a gauge field  $A_\mu$  in  $\mathfrak{h}$  and define a covariant derivative

$$D_\mu g = \partial_\mu g - g A_\mu \tag{1.1}$$

with the property that

$$g \mapsto gh, \quad A_\mu \mapsto h^{-1} A_\mu h + h^{-1} \partial_\mu h \quad \Rightarrow \quad D_\mu g \mapsto (D_\mu g)h \tag{1.2}$$

for any function  $h(x^\mu)$ . It is also useful to introduce the  $\mathfrak{g}$ -valued currents

$$j_\mu = g^{-1} \partial_\mu g, \quad J_\mu = g^{-1} D_\mu g = j_\mu - A_\mu. \tag{1.3}$$

The latter current is covariant under gauge transformations, with

$$J_\mu \mapsto h^{-1} J_\mu h . \quad (1.4)$$

The  $G/H$  sigma-model is defined by the lagrangian

$$\mathcal{L} = -\frac{1}{2}\text{Tr}(J^\mu J_\mu) = -\frac{1}{2}\text{Tr}(g^{-1} D^\mu g g^{-1} D_\mu g) \quad (1.5)$$

which has a global  $G$  symmetry (acting from the left on  $g$ ) and the local  $H$  symmetry discussed above. The equation of motion for the field  $g$  is

$$D_\mu J^\mu = \partial_\mu J^\mu + [A_\mu, J^\mu] = 0 . \quad (1.6)$$

By combining this with the identity  $\partial_\mu j_\nu - \partial_\nu j_\mu + [j_\mu, j_\nu] = 0$  we obtain

$$2\partial_- J_+ + 2[A_-, J_+] = [J_+, J_-] + F_{+-} \quad (1.7)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ , and we have made use of light-cone coordinates,  $V_\pm = V_0 \pm V_1$ . The equations of motion for the  $A_\mu$  fields are

$$J_\mu = 0 \quad \text{on } \mathbf{h} . \quad (1.8)$$

Everything we have said so far applies to an arbitrary homogeneous space  $G/H$ . The special nature of symmetric spaces emerges when we look for conserved quantities. It is helpful to recall what happens for the PCM based on  $G$ , which can be obtained by specializing the analysis above to the case in which  $H$  is trivial, setting  $A_\mu = 0$ , and hence  $J_\mu = j_\mu$ . The equation of motion then becomes  $\partial_- j_+ = \frac{1}{2}[j_+, j_-]$  which implies conservation equations such as  $\partial_- \text{Tr}(j_+^m) = 0$ . For a general, non-trivial gauge group  $H$ , the additional term  $F_{+-}$  in (1.7) prevents one from carrying out a similar construction in any obvious way. But for  $G/H$  a symmetric space there is an orthogonal decomposition of the Lie algebra, and a compatible  $\mathbf{Z}_2$  grading of the Lie bracket,

$$\mathbf{g} = \mathbf{h} + \mathbf{k} : \quad [\mathbf{h}, \mathbf{h}] \subset \mathbf{h}, \quad [\mathbf{h}, \mathbf{k}] \subset \mathbf{k}, \quad [\mathbf{k}, \mathbf{k}] \subset \mathbf{h} . \quad (1.9)$$

Since the  $A_\mu$  equations of motion force  $J_\mu$  to take values in  $\mathbf{k}$ , this grading then implies that the left- and right-hand sides of (1.7) must vanish separately:

$$\partial_- J_+ = -[A_-, J_+] \quad \text{in } \mathbf{k}, \quad [J_+, J_-] = -F_{+-} \quad \text{in } \mathbf{h} . \quad (1.10)$$

The first of these conditions allows the construction of conservation laws very similar to those of the PCM, *e.g.*  $\partial_- \text{Tr}(J_+^m) = 0$ . A conserved current of this type can be written down using any symmetric invariant tensor, and we now discuss the possibilities.

## 2 Currents and invariants on $G/H$

Let us introduce a basis of anti-hermitian generators  $\{t^a\}$  for  $\mathfrak{g}$ , obeying  $[t^a, t^b] = f^{abc}t^c$  and  $\text{Tr}(t^a t^b) = -\delta^{ab}$ . Since  $\mathfrak{g}$  is compact, we need not distinguish upper and lower Lie algebra indices, and the structure constants  $f^{abc}$  are real and totally antisymmetric. We can assume our basis is chosen so that it splits  $a \rightarrow (\hat{\alpha}, \alpha)$ , with the subsets  $\{t^{\hat{\alpha}}\}$  and  $\{t^{\alpha}\}$  providing bases for  $\mathfrak{h}$  and  $\mathfrak{k}$  respectively in the decomposition in (1.9). The currents of the  $G/H$  SSM model can now be written  $j_{\mu}^a$ , while the non-trivial components of the gauge fields are  $A_{\mu}^{\hat{\alpha}}$ . The symmetric space condition, or  $\mathbf{Z}_2$  grading, means that the non-vanishing structure constants are  $f_{\hat{\alpha}\hat{\beta}\hat{\gamma}}$ , and  $f_{\hat{\alpha}\beta\gamma}$ , up to permutations of indices.

Consider some totally symmetric tensor  $d_{a_1 \dots a_m}^{(m)}$  of degree  $m$  on  $\mathfrak{g}$  (we use ‘degree’ rather than ‘rank’ to avoid confusion with the rank of the algebra; we shall not always indicate the degree explicitly). By virtue of its symmetry, this tensor is completely determined by the associated function  $d(X) \equiv d_{a_1 \dots a_m} X^{a_1} \dots X^{a_m}$  where  $X = X^a t^a$  is an arbitrary element of the Lie algebra. If  $\tau$  is any map from  $\mathfrak{g}$  to itself, then we define a new tensor  $\tau(d)$  by  $\tau(d)(X) = d(\tau(X))$ . We shall call  $d$  a  $G$ -invariant tensor, or simply an *invariant tensor on  $\mathfrak{g}$* , if  $d = \tau(d)$  whenever  $\tau$  is an inner automorphism of  $\mathfrak{g}$ , so that  $\tau(X) = gXg^{-1}$  for some  $g$  in  $G$ . This is equivalent to the condition

$$d_{c(a_1 \dots a_{m-1} f_a)bc}^{(m)} = 0 .$$

Similar definitions apply in an obvious way to subgroups of  $G$  acting on subspaces of  $\mathfrak{g}$ .

A conserved charge of spin  $s$  in the PCM based on  $G$  can be constructed from each symmetric invariant tensor  $d_{a_1 a_2 \dots a_s a_{s+1}}^{(s+1)}$  [6]. A related conservation law in the  $G/H$  SSM arises by restricting such a tensor to  $\mathfrak{k}$ , thus considering just the components  $d_{\alpha_1 \alpha_2 \dots \alpha_s \alpha_{s+1}}^{(s+1)}$ . The invariance condition written above means that the restricted tensor obeys

$$d_{\gamma(\alpha_1 \dots \alpha_s f_{\alpha})\beta\gamma}^{(s+1)} = 0 . \quad (2.1)$$

It is easy to check directly that this implies

$$\partial_{-}(d_{\alpha_1 \dots \alpha_{s+1}} J_{+}^{\alpha_1} \dots J_{+}^{\alpha_{s+1}}) = 0 \quad (2.2)$$

on using the equation of motion in (1.10). From this current we obtain a conserved charge of spin  $s$  in the usual way:

$$q_s = \int dx d_{\alpha_1 \dots \alpha_{s+1}} J_{+}^{\alpha_1} \dots J_{+}^{\alpha_{s+1}} . \quad (2.3)$$

Similar conserved charges can be constructed from  $J_-^\alpha$ ; their properties are directly analogous and we will not discuss them further.

It is important to check that the tensor  $d^{(s+1)}$  does not vanish when restricted to  $\mathbf{k}$  in order to have a *non-trivial* conserved quantity in the SSM. We shall return to this point in section 6. It is also clear that (2.1) and (2.2) rely only on the fact that the tensor is  $H$ -invariant on  $\mathbf{k}$ . It is not obvious, a priori, that any such tensor should arise as the restriction of some  $G$ -invariant tensor on  $\mathbf{g}$ , but this emerges from the analysis below. These and other matters can be understood in terms of the special role played by *primitive* invariants.

For each Lie algebra  $\mathbf{g}$  there are exactly  $\text{rank}(G)$  primitive symmetric invariants, with the property that all others can be written as polynomial functions of them (see *e.g.* [9]). This happens essentially because any element in  $\mathbf{g}$  is conjugate to some element in a fixed Cartan subalgebra (CSA), and so any invariant tensor is determined by its values on the  $\text{rank}(G)$  independent basis elements of this CSA. The degrees of the primitive invariant tensors for each classical group  $G$  are given in the table, in terms of the exponents,  $s$ .

One convenient choice for the primitive invariants consists of symmetric traces, with  $d(X) = \text{Tr}(X^m)$  (of the appropriate degrees), together with the Pfaffian invariant  $d(X) = \text{Pf}(X) \equiv \epsilon_{i_1 j_1 \dots i_n j_n} X_{i_1 j_1} \dots X_{i_n j_n}$  for the special case of  $SO(2n)$ . Other choices are possible, and will be important later. However, any choices of the primitive invariants differ only by terms which are products of polynomials of lower degrees.

We need to determine how these facts generalize from a group  $G$  to a symmetric space  $G/H$ . One can define a CSA for  $G/H$  as a maximal set of mutually commuting generators in  $\mathbf{k}$ , and  $\text{rank}(G/H)$  is then the number of elements in such a set. It can also be shown that any element of  $\mathbf{k}$  is conjugate, by an element of  $H$ , to a member of some chosen CSA [10]. This means that, just as for groups, any invariant is determined by its values on the CSA, and there are precisely  $\text{rank}(G/H)$  primitive invariants, in terms of which all others can be expressed.

The degrees of the primitive invariants for each symmetric space  $G/H$  with  $G$  classical are given by the data in the table. Each primitive  $H$ -invariant tensor is obtained by restricting a primitive  $G$ -invariant tensor on  $\mathbf{g}$  to the subspace  $\mathbf{k}$ . Other invariants which are primitive on  $\mathbf{g}$  may vanish when restricted to  $\mathbf{k}$ , or else fail to be primitive on  $\mathbf{k}$  in some more complicated fashion. For our purposes we may take the values  $s$  given in the table

as a definition of the exponents of  $G/H$ . The Pfaffian in  $SO(2n)$  appears as something of a special case, lying outside the regular sequence formed by the other invariants. Because of this, we have chosen to separate the values of  $s$  corresponding to the Pfaffian, or its restriction, by a semi-colon.

$G/H$ symmetric space	$\text{rank}(G/H)$	$s : d^{(s+1)}$ primitive	$h$
$SU(n)$	$n-1$	$1, 2, \dots, n-1$	$n$
$SO(2n+1)$	$n$	$1, 3, \dots, 2n-1$	$2n$
$SO(2n)$	$n$	$1, 3, \dots, 2n-3; n-1$	$2n-2$
$Sp(2n)$	$n$	$1, 3, \dots, 2n-1$	$2n$
$SU(p+q)/S(U(p) \times U(q)) \quad (p \leq q)$	$p$	$1, 3, \dots, 2p-1$	$2p$
$SO(p+q)/SO(p) \times SO(q) \quad (p < q)$	$p$	$1, 3, \dots, 2p-1$	$2p$
$SO(2n)/SO(n) \times SO(n)$	$n$	$1, 3, \dots, 2n-3; n-1$	$2n-2$
$Sp(2p+2q)/Sp(2p) \times Sp(2q) \quad (p \leq q)$	$p$	$1, 3, \dots, 2p-1$	$2p$
$SU(n)/SO(n)$	$n-1$	$1, 2, \dots, n-1$	$n$
$Sp(2n)/U(n)$	$n$	$1, 3, \dots, 2n-1$	$2n$
$SO(2n)/U(n)$	$[n/2]$	$1, 3, \dots, 2[n/2]-1$	$2[n/2]$
$SU(2n)/Sp(2n)$	$n-1$	$1, 2, \dots, n-1$	$n$

We obtained the results in the table by making use of convenient canonical forms for the CSA generators in  $G/H$ . In view of the remarks above, it is the set of eigenvalues of the CSA generators, and how they behave under  $H$ , which determines the allowed invariants, and which of them are primitive. By comparing with the well-known data for classical groups, it is not difficult to arrive at the results for symmetric spaces. The method is best illustrated by some examples, but a fuller explanation of this sort requires some additional technical preparation, and so we consign these details to a separate section below. Our main task—understanding the conserved charges in the  $G/H$  sigma-model—will then be resumed in section 4.

### 3 Some details and examples

Recall that the CSA of each classical Lie algebra can be parameterized by a set of real ‘eigenvalues’  $\lambda_i$  as follows

$$\begin{aligned} su(n) : & \quad \text{diag}(i\lambda_1, \dots, i\lambda_n) \quad \text{with} \quad \lambda_1 + \dots + \lambda_n = 0 \\ so(2n) : & \quad \text{diag}\left(\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{bmatrix}\right) \\ so(2n+1) : & \quad \text{diag}\left(\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{bmatrix}, 0\right) \\ sp(2n) : & \quad \text{diag}(i\lambda_1, \dots, i\lambda_n, -i\lambda_1, \dots, -i\lambda_n) \end{aligned}$$

where we have used an obvious block notation for the orthogonal algebras. The function  $d(X)$  defined by a symmetric invariant tensor  $d$  on  $\mathfrak{g}$  is some polynomial in the eigenvalues  $\lambda_i$ . This polynomial must be totally symmetric, because in all cases a member of the CSA of  $\mathfrak{g}$  can be conjugated by specific elements of  $G$  so as to permute the eigenvalues in any desired way. (To show this it actually suffices to use the simple result  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$  in appropriate block forms.)

Now, for  $\mathfrak{g} = su(n)$  there are invariant tensors corresponding to all symmetric polynomials in the eigenvalues, except for  $\lambda_1 + \dots + \lambda_n = 0$ . The symmetrized traces mentioned earlier clearly give rise to the power sums  $\sum_i \lambda_i^m$ , and a basis of primitive invariants corresponds to the finite subset with  $m = s+1$  and  $s$  an exponent. On the other hand, for  $so(2n+1)$  and  $sp(2n)$ , only polynomials in *even* powers of  $\lambda_i$  are allowed, since the sign of any eigenvalue can be reversed by conjugating with a suitable element of  $G$ . For example, conjugating a CSA element of  $\mathfrak{g} = so(2n+1)$  with  $\text{diag}(1, -1, 0, \dots, 0, -1)$  (which

certainly belongs of  $G = SO(2n+1)$ ) changes the sign of  $\lambda_1$ . For  $\mathfrak{g} = so(2n)$  it is also possible to reverse the signs of eigenvalues, but only in *pairs*. For instance, we can conjugate by  $\text{diag}(1, -1, 1, -1, 0, \dots, 0)$  which reverses the signs of  $\lambda_1$  and  $\lambda_2$ , but we cannot conjugate by any element of  $G = SO(2n)$  and change the sign of just one eigenvalue. In addition to the even symmetric powers, these symmetry properties allow precisely one more independent invariant, which is the Pfaffian, proportional to  $\lambda_1 \dots \lambda_n$ .

In this manner the allowed invariants and primitive invariants for each Lie algebra are characterized as certain totally symmetric polynomials in the CSA eigenvalues. Moreover, we can distinguish three classes of polynomials, depending on their additional symmetry properties: A-type, like  $su(n)$ —no additional symmetries; B/C-type, like  $so(2n+1)$  or  $sp(2n)$ —invariant under reversal of sign of each eigenvalue separately; D-type, like  $so(2n)$ —invariant under reversal of signs of pairs of eigenvalues.<sup>3</sup>

The results for each symmetric space  $G/H$  can now be found by the following steps. First, find a convenient parameterization of  $\mathfrak{k}$  and its CSA, as in *e.g.* [10];  $H$ -invariant tensors on  $\mathfrak{k}$  are polynomials in the ‘eigenvalues’ of the CSA generators. Next, examine the action (via conjugation) of specific elements of  $H$  on these eigenvalues, and so determine that the polynomials have symmetry type A, B/C, or D using the terminology introduced above. Finally, check that all primitive invariants of this type indeed arise as restrictions of  $G$ -invariant tensors on  $\mathfrak{g}$ , *e.g.* by considering traces or symmetric powers. We now sketch how this works for some examples; the remaining cases in the table can be handled similarly.

The Grassmannians  $SO(p+q)/SO(p) \times SO(q)$  and  $SU(p+q)/S(U(p) \times U(q))$  are conveniently treated together. For either family the subgroup  $H \subset G$  has the block structure  $\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$  and  $\mathfrak{k}$  consists of matrices  $\begin{pmatrix} 0 & X \\ -X^\dagger & 0 \end{pmatrix}$  where  $P$  is  $(p \times p)$ ,  $Q$  is  $(q \times q)$ , and  $X$  is  $(p \times q)$  with real or complex entries. For both the real and complex families, the CSA can be parameterized by

$$X = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_p & 0 & \cdots & 0 \end{pmatrix}$$

with real ‘eigenvalues’  $\lambda_i$  (we assume  $p \leq q$ ). Notice that the effect on  $\mathfrak{k}$  of conjugating by a general element of  $H$  is  $X \mapsto PXQ^{-1}$ . One can readily choose  $P$  and  $Q$  so as to permute

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<sup>3</sup> These symmetry operations on the CSA eigenvalues are usually presented as actions of the Weyl group.



the CSA eigenvalues in any desired way.

When  $p < q$ , it is also easy to find elements of  $H$  which change the sign of any given eigenvalue, just as in the previous discussion of  $so(2n+1)$ . This implies that the invariant polynomials are of B/C-type. They arise as restrictions of symmetric traces on  $\mathfrak{g}$ , and in fact they can be written  $\text{Tr}(XX^\dagger \dots XX^\dagger)$ , which is manifestly invariant under  $H$ . When  $p = q$ , a new feature arises for the real Grassmannians: just as for the Lie algebra  $so(2p)$ , it is now only possible to change the signs of eigenvalues in pairs, so the pattern of invariants is type D. An additional primitive invariant on  $\mathfrak{k}$  arises as the restriction of the Pfaffian on  $\mathfrak{g} = so(2p)$ , and it can be written  $\epsilon_{i_1 \dots i_p} \epsilon_{j_1 \dots j_p} X_{i_1 j_1} \dots X_{i_p j_p}$ . Note that under the action of  $H$  this expression changes by a factor  $\det(P) \det(Q^{-1})$  which is indeed unity for these examples. For the complex Grassmannians, however, the invariants are still of type B/C, even when  $p = q$ . This is because there are elements in  $H$  such as  $P = Q^{-1} = \text{diag}(i, 1, \dots, 1)$  which change the sign of a single eigenvalue, in this case  $\lambda_1$ . Consistent with this, such elements have  $\det(P) \det(Q^{-1}) \neq 1$ .

The next example is  $G/H = SU(n)/SO(n)$ . The subspace  $\mathfrak{h}$  consists of real, anti-symmetric matrices, while  $\mathfrak{k}$  consists of imaginary, symmetric, traceless matrices, and its natural CSA coincides with the standard choice for  $su(n)$ . Invariance under  $H = SO(n)$  means precisely that the eigenvalues can be permuted in any desired way. The set of invariants is of type A, the same as those for  $\mathfrak{g} = su(n)$ , and they clearly descend from these by restriction to  $\mathfrak{k}$ .

Finally, consider  $G/H = SU(2n)/Sp(2n)$ . We can choose the subspaces  $\mathfrak{h}$  and  $\mathfrak{k}$  to consist of matrices with the block forms  $\begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}$  and  $\begin{pmatrix} C & D \\ D^* & -C^* \end{pmatrix}$  respectively, where  $A$  lives in  $u(n)$ ,  $B$  is complex and symmetric,  $C$  lives in  $su(n)$ , and  $D$  is complex and antisymmetric. The CSA elements are  $X = \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix}$  where  $Y$  is any generator in the standard CSA of  $su(n)$ . The set of primitive invariants is once again A-type, matching those of  $su(n)$ . Unlike in the previous example, however, they arise here as restrictions of tensors on  $\mathfrak{g} = su(2n)$ .

## 4 Poisson brackets of currents and charges

Returning now to our treatment of the  $G/H$  sigma-model, we are interested in the classical Poisson brackets (PBs) of the currents  $j_\mu^a$  and  $J_\mu^a$ . A convenient way to calculate these

directly, without first finding PBs for the underlying fields  $g$ , is to use the approach of [11], which is easily adapted to the present situation. We take  $j_1^a$  as an independent canonical coordinate and view  $j_0^a$  as a non-local function of it by means of the identity  $\partial_0 j_1 - \partial_1 j_0 + [j_0, j_1] = 0$ , which implies

$$j_0 = \mathcal{D}^{-1}(\partial_0 j_1) \quad \text{where} \quad \mathcal{D}X \equiv \partial_1 X + [j_1, X] . \quad (4.1)$$

(We must assume appropriate boundary conditions on the fields, to allow the definition and manipulation of inverse differential operators.) In addition, the field  $A_1^{\hat{\alpha}}$  can be eliminated from the lagrangian immediately by its equation of motion,  $J_1^{\hat{\alpha}} = j_1^{\hat{\alpha}} - A_1^{\hat{\alpha}} = 0$ . On making these replacements, the lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\mathcal{D}^{-1}\partial_0 j_1 - A_0)^a (\mathcal{D}^{-1}\partial_0 j_1 - A_0)^a - \frac{1}{2}j_1^\alpha j_1^\alpha . \quad (4.2)$$

The momentum conjugate to  $j_1^a$  is

$$\pi^a = -(\mathcal{D}^{-2}\partial_0 j_1 - \mathcal{D}^{-1}A_0)^a \quad (4.3)$$

and we impose canonical, equal-time PBs

$$\{j_1^a(x), \pi^b(y)\} = \delta^{ab}\delta(x-y) . \quad (4.4)$$

The resulting Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}J_0^a J_0^a + A_0^{\hat{\alpha}} J_0^{\hat{\alpha}} + \frac{1}{2}j_1^\alpha j_1^\alpha \quad (4.5)$$

where

$$J_0^a = -\mathcal{D}\pi^a . \quad (4.6)$$

The remaining, time-like component of the gauge field  $A_0^{\hat{\alpha}}$  acts as a Lagrange multiplier, imposing the constraint

$$J_0^{\hat{\alpha}} \approx 0 . \quad (4.7)$$

(We have taken a well-known short-cut by applying Dirac's procedure [12] without introducing momenta conjugate to the gauge fields, which play the role of Lagrange multipliers.)

To summarize: the independent canonical variables are  $j_1^a$  and  $\pi^a$ , obeying (4.4). In terms of these,  $j_0^a$  is defined by (4.1);  $J_1^{\hat{\alpha}} = 0$ ;  $J_1^\alpha = j_1^\alpha$ ; and  $J_0^a$  is defined by (4.6). The system is governed by the Hamiltonian density in (4.5), together with the constraint (4.7). From (4.4) it is straightforward to calculate the equal time PBs

$$\begin{aligned} \{J_0^a(x), J_0^b(y)\} &= -f^{abc} J_0^c(x) \delta(x-y) \\ \{J_0^a(x), j_1^b(y)\} &= -f^{abc} j_1^c(x) \delta(x-y) + \delta^{ab} \delta'(x-y) \\ \{j_1^a(x), j_1^b(y)\} &= 0 \end{aligned} \quad (4.8)$$

which are the objects of central importance for us. The first of these implies that the constraints (4.7) are first-class, corresponding to the original  $H$  gauge invariance. (They are analogous to the Gauss Law constraint arising in canonical treatments of electromagnetism.) It is also a simple consequence of (4.8) that the constraints weakly commute with the Hamiltonian, so they are preserved in time, and the canonical formulation has therefore been completed in a consistent fashion.

Let us now consider the canonical brackets of two conserved charges of type (2.3). We need not concern ourselves with gauge fixing the remaining constraints (4.7), because each current (2.2) commutes with them (each current is invariant under  $H$ -gauge transformations) and so the Dirac bracket of two conserved charges, obtained after imposing some gauge choice, is always identical to their Poisson bracket

$$\{q_s, q_r\} = \left\{ \int dx d_{\alpha_1 \dots \alpha_{s+1}}^{(s+1)} J_+^{\alpha_1}(x) \dots J_+^{\alpha_{s+1}}(x), \int dy d_{\beta_1 \dots \beta_{r+1}}^{(r+1)} J_+^{\beta_1}(y) \dots J_+^{\beta_{r+1}}(y) \right\}$$

which can be calculated from (4.8). The terms in (4.8) containing  $\delta(x-y)$  do not contribute, by invariance of the  $d$ -tensors (the arguments are exactly similar to those given in [6] for the PCM). The terms involving  $\delta'(x-y)$ , on the other hand, result in an integrand proportional to

$$d_{\alpha_1 \dots \alpha_s \gamma}^{(s+1)} d_{\beta_1 \dots \beta_r \gamma}^{(r+1)} J_+^{\alpha_1} \dots J_+^{\alpha_s} J_+^{\beta_1} \dots J_+^{\beta_r} \partial_1 J_+^{\beta_r}.$$

The charges  $q_s$  and  $q_r$  will commute if and only if this integrand is a total derivative, which is true if and only if

$$d_{(\alpha_1 \dots \alpha_s}^{(s+1)} \gamma d_{\beta_1 \dots \beta_{r-1}) \beta_r}^{(r+1)} \gamma = d_{(\alpha_1 \dots \alpha_s}^{(s+1)} \gamma d_{\beta_1 \dots \beta_r)}^{(r+1)} \gamma. \quad (4.9)$$

It was shown in [6] (see also eqn. (2.39) of [13]) that there exist tensors  $k^{(s+1)}$  for each classical Lie group  $G$  with the property that

$$k_{(a_1 \dots a_s}^{(s+1)} c k_{b_1 \dots b_{r-1}) b_r}^{(r+1)} c = k_{(a_1 \dots a_s}^{(s+1)} c k_{b_1 \dots b_r)}^{(r+1)} c. \quad (4.10)$$

This is exactly what is required to ensure commuting charges in the PCM based on  $G$ , and these tensors exist whenever  $s$  is an exponent of  $G$  modulo  $h$ . An obvious possibility is to choose the same tensors  $k^{(s+1)}$  for  $G/H$  as for  $G$ . We can certainly restrict the free indices in (4.10) from  $\mathbf{g}$  to the subspace  $\mathbf{k}$ , but it is not clear that we can restrict the repeated index from  $c$  in (4.10) to  $\gamma$ , so as to arrive at (4.9). Such a restriction is allowed in all cases of interest, however, by virtue of the property

$$d_{\alpha_1 \dots \alpha_s \alpha_{s+1}}^{(s+1)} \neq 0 \quad \Rightarrow \quad d_{\alpha_1 \dots \alpha_s \hat{\gamma}}^{(s+1)} = 0. \quad (4.11)$$

We shall prove in the next section that this holds for any invariant tensor on a classical symmetric space.

## 5 More properties of invariant tensors on $G/H$

Let us return to the definition of  $G/H$  via a  $\mathbf{Z}_2$  grading of  $\mathfrak{g}$ . An equivalent statement is that there is a Lie algebra automorphism  $\sigma$  of  $\mathfrak{g}$  with  $\sigma^2 = 1$ . The subspaces  $\mathfrak{h}$  and  $\mathfrak{k}$  are the eigenspaces of  $\sigma$  with eigenvalues  $+1$  and  $-1$  respectively.

An invariant tensor  $d^{(m)}$  on  $\mathfrak{g}$  is not necessarily invariant under  $\sigma$  (unless  $\sigma$  is an inner automorphism of  $\mathfrak{g}$ ), but let us assume for the moment that  $\sigma(d) = d$ . Since elements of  $\mathfrak{k}$  change sign under  $\sigma$ , it follows that if the degree  $m$  is odd, then  $d^{(m)}$  must vanish when restricted to  $\mathfrak{k}$ . If  $m$  is even,  $d^{(m)}$  need not vanish on  $\mathfrak{k}$ , but then it must satisfy  $d_{\alpha_1 \dots \alpha_{m-1} \hat{\gamma}}^{(m)} = 0$ , since  $m-1$  is odd.

Now suppose that  $d$  is not invariant under  $\sigma$ . It turns out that in all such cases,  $d$  is instead invariant under the map  $\tilde{\sigma}(X) \equiv -\sigma(X)$ , as we shall see below. The map  $\tilde{\sigma}$  is not an automorphism of  $\mathfrak{g}$ , and it has eigenspaces  $\mathfrak{h}$  and  $\mathfrak{k}$  with the reversed eigenvalues  $-1$  and  $+1$  respectively. Requiring that  $d^{(m)}$  be invariant under  $\tilde{\sigma}$  and that it be non-zero on  $\mathfrak{k}$  does not result in any restriction on the degree  $m$ . However, since  $\tilde{\sigma}$  reverses the sign of each element in  $\mathfrak{h}$ , invariance of  $d$  under this map does imply that  $d_{\alpha_1 \dots \alpha_{m-1} \hat{\gamma}}^{(m)} = 0$ , as required.

We conclude that (4.11) holds for every tensor  $d$  obeying either  $\sigma(d) = d$ , or  $\tilde{\sigma}(d) = d$ . To check that this exhausts all possibilities, we can consider the simple explicit forms for  $\sigma$  that are available for each of the classical symmetric spaces [10].

For the three families of Grassmannian symmetric spaces as well as for  $SO(2n)/U(n)$  and  $Sp(2n)/U(n)$ , we can take  $\sigma$  to be a map on  $\mathfrak{g}$  of the form

$$\sigma(X) = MXM^{-1} ,$$

for some matrix  $M$ . (Once again, this is not necessarily an inner automorphism, because  $M$  may not belong to  $G$ .) Clearly this implies  $\sigma(d) = d$  whenever  $d$  is a symmetrized trace. The only primitive invariant not of this type is the (restriction of) the Pfaffian for  $SO(2n)/SO(n) \times SO(n)$ . In this case  $\sigma(d) = (\det M)d = (-1)^n d$ , since  $M$  is diagonal with  $n$  pairs of eigenvalues  $\pm 1$ . When  $n$  is even,  $\sigma(d) = d$ , but when  $n$  is odd, the degree of  $d$  is also odd, and then  $\tilde{\sigma}(d) = d$ , as claimed.<sup>4</sup>

Finally we consider the families  $SU(n)/SO(n)$  and  $SU(2n)/Sp(2n)$  for which the auto-

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<sup>4</sup> These remarks are consistent with the patterns of invariants given in earlier sections: some cosets always have  $\sigma(d) = d$ , and their invariants were already found to have even degrees.

morphism of  $\mathfrak{g}$  can be written in the form

$$\sigma(X) = MX^*M^{-1} \quad \Rightarrow \quad \tilde{\sigma}(X) = MX^T M^{-1},$$

since  $X$  is anti-hermitian. For the first family  $M$  is the identity, while for the second it is some standard symplectic structure. In either case it is clear that  $\tilde{\sigma}(d) = d$  whenever  $d$  is a symmetrized trace, and it follows that the same is true for any tensor  $d$ .

## 6 The classical spectrum of commuting charges

Our results concerning commuting charges in the  $G/H$  SSM can now be summarized as follows. From each invariant tensor  $k^{(s+1)}$  on  $\mathfrak{g}$  we obtain a conserved current in the  $G/H$  model by restricting the tensor to  $\mathbf{k}$ , leading to a conserved charge of spin  $s$ . These conserved charges always commute with one another, by virtue of the arguments given in sections 4 and 5 above. The final issue we must settle is precisely which tensors  $k^{(s+1)}$  are non-zero when restricted to  $\mathbf{k}$ , so as to give non-trivial conserved charges for the  $G/H$  SSM.

Let us first recall what happens for a group  $G$  [6]. The tensors  $k^{(s+1)}$  provide a set of primitive invariants when  $s$  runs over the exponents of  $G$ . These primitive invariants have  $s < h$ , the Coxeter number of  $G$ . For larger values of  $s$ , the tensors  $k^{(s+1)}$  are not primitive, but they are still non-zero (yielding non-trivial commuting charges) precisely when  $s$  is equal to an exponent of  $G$  modulo  $h$ .

We have discussed in detail in sections 3 and 4 the patterns of invariants for each  $G/H$ , and how these are obtained from invariants on  $G$ . We can certainly choose our primitive tensors on  $G/H$  from amongst the  $k^{(s+1)}$  required for commuting charges. We have already defined the corresponding values of  $s$  to be the exponents of the symmetric space  $G/H$ . Similarly, we now define an integer  $h$  for each  $G/H$ , with values specified in the table; this will play the role of the Coxeter number.<sup>5</sup>

With these definitions, our final result can be stated very simply:  $k^{(s+1)}$  is non-zero on  $\mathbf{k}$  only when  $s$  is equal to an exponent of  $G/H$  modulo  $h$ . Thus, just as for groups, there are commuting charges associated with primitive invariants and hence exponents  $s < h$ ,

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<sup>5</sup>The discussion in section 3 effectively associates a Weyl group to each symmetric space, and our definition of  $h$  coincides with the standard one for such groups [7].

and also with non-primitive, but non-vanishing invariants for values of  $s > h$  which repeat in families modulo  $h$ .

In some cases this result is automatic, because the values of  $s$  run over all the positive odd integers, which trivially repeat modulo an even value of  $h$ . In other instances the pattern of spins is more involved, but the result for  $G/H$  still follows directly from the corresponding statement for the group  $G$ . However, there is one family, namely  $SU(2n)/Sp(2n)$ , which requires special care. For these spaces we have defined  $h = n$ , and so the validity of our result depends, in particular, upon the fact that the tensors  $k^{(pn+1)}$  should vanish when restricted to  $\mathbf{k}$  for any integer  $p$ . When  $p$  is even we know that this tensor actually vanishes on  $\mathbf{g}$ , but when  $p$  is odd it is non-zero on  $\mathbf{g}$  and it is not immediately clear why it should vanish on  $\mathbf{k}$ .

To understand how this comes about we need to look more closely at the definition of the  $k$ -tensors for the algebras  $su(n)$ , and to make clear which algebra we are talking about we shall write  $k_{su(n)}^{(m)}$ . Although these objects depend upon the value of  $n$ , they do so in a rather simple way, as revealed by the formula [6]:

$$k_{su(n)}^{(m)}(X) = \mathcal{A}^{(m)} \left( \frac{1}{n} \text{Tr}(X^m), \frac{1}{n} \text{Tr}(X^{m-1}), \dots, \frac{1}{n} \text{Tr}(X^2) \right) .$$

The polynomials  $\mathcal{A}^{(m)}$  are in fact characterized by the property  $k_{su(n)}^{(n+1)} = 0$ , and it can be shown that this implies  $k_{su(n)}^{(pn+1)} = 0$  for any integer  $p$  [6].

The coset  $SU(2n)/Sp(2n)$  was one of the examples discussed earlier in section 4. The CSA can be chosen to consist of matrices of the form  $X = \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix}$  where  $Y$  is diagonal, traceless, and pure-imaginary, and hence belongs to the CSA of  $su(n)$ . Now clearly

$$\begin{aligned} k_{su(2n)}^{(m)}(X) &= \mathcal{A}^{(m)} \left( \frac{1}{2n} \text{Tr}(X^m), \frac{1}{2n} \text{Tr}(X^{m-1}), \dots, \frac{1}{2n} \text{Tr}(X^2) \right) \\ &= \mathcal{A}^{(m)} \left( \frac{1}{n} \text{Tr}(Y^m), \frac{1}{n} \text{Tr}(Y^{m-1}), \dots, \frac{1}{n} \text{Tr}(Y^2) \right) \\ &= k_{su(n)}^{(m)}(Y) . \end{aligned}$$

The last expression vanishes when  $m = pn+1$  for any integer  $p$ , as mentioned above. Since any invariant tensor is determined by its values on a CSA, we conclude that  $k_{su(2n)}^{(pn+1)}$  indeed vanishes when restricted to  $\mathbf{k}$ , as claimed.

## 7 Comments

The results of this paper reinforce the elegant mathematical structure underlying the classical integrability of sigma-models on symmetric spaces. It would be interesting to investigate whether the charges we have constructed might be related to those arising in [1], whose properties are rather more mysterious. There are also a number of other directions for future work. Although we have considered only classical groups, we expect similar results to hold for the exceptional groups and their symmetric spaces. At the quantum level, integrability is believed to depend upon  $H$  being simple, and for these cases exact S-matrices have been proposed [14]. It would be interesting to examine whether quantum-mechanical survival of our charges is consistent with these proposals, in light of the treatment of affine Toda theories in [8, 7]. Finally, one could extend the results of [13] to supersymmetric  $G/H$  models, with the added incentive that these are believed to be quantum-integrable for any symmetric space [15].

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